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A decidability result of the logic based on fuzzy rough sets

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Abstract

Rough sets, which correspond to the approximation of sets by means of equivalence relations modeling indiscernibility, are extended to fuzzy sets and fuzzy equivalence relations. This paper proposes a logic based on the fuzzy rough sets. Then, it is shown that the satisfiability problem for this logic is solvable.

§1 Introduction

In [1], Z. Pawlak introduced rough sets, which correspond to the approximation of sets by means of equivalence relations modeling indiscernibility. In [2], E. Orlowska and Z. Pawlak discussed theoretical foundations of knowledge representation from a point of view of modal logic. Further, in [3] E. Orlowska presented a logic of indiscernibility relations.

Recently, the problem of knowledge representation has been one of the central topics of artificial intelligence. Along this line, in [4] the present author has given a survey of relationships between various non-standard logics and knowledge representation. Indiscernibility relations are primarily based on equivalence relations and thus its logic corresponds to the modal logic S5.

Meanwhile, the theory of fuzzy sets introduced in L.A. Zadeh [5] is a well-established and active field, with numerous applications. In [6], the author introduced fuzzy rough sets based on extended equivalence relations that correspond to the similarity proposed in Zadeh [7].

The purpose of this paper is to propose a logic based on the fuzzy rough sets and some relevant notions and then to show that the satisfiability problem for this logic is solvable. We assume that readers are familiar with primitive definitions of fuzzy rough sets [6]. This paper is partly motivated Dubois and Prade [8].

§2 Terminologies

Let us consider a system S=(OB, AT, VAL, f). In this system S, OB, AT, VAL and f are defined as follows:

OB is a set (not necessary finite) of objects,

At is a set (not necessary finite) of attributes,

VAL_a is a set of values of attribute a for each $a \in AT$, and VAL is the union of all the sets VAL_a,

f is a mapping from $OB \times AT$ into VAL.

We consider a subset A of AT and a relation $\tilde{A}$ in the set OB defined as follows:

$$(o_i, o_j) \in \tilde{A} \quad \text{iff } f(o_i, a) = f(o_j, a) \text{ for each } a \in A.$$

$$\tilde{\phi} = OB \times OB.$$

Relation $\tilde{A}$ is refered to as *indiscernibility* with respect to attributes for the set A.

Let R be an equivalence relation defined on X. $[x]_R$ stands for the equivalence class of x in the sense of R.

Definition 2.1 Let S be a subset of a given set X and R be an equivalence relation defined on X.

$R_*(S)=\{x\in X \mid [x]_R \subseteq S\}$,

$R^*(S)=\{x\in X \mid [x]_R \cap S \neq \phi\}$.

$R_*(S)$ and $R^*(S)$ are called a *lower* and an *upper approximation* of S with respect to R, respectively.

Definition 2.2 A rough set is a collection of objects which cannot be precisely characterized in terms of the values of a set of attributes, while a lower and an upper approxiomation of the collection can be characterized in terms of these attributes.

Now, let us consider the fuzzy case of the previous definitions. Fuzzy sets are defined by their membership function μ [5]. Let S and T be fuzzy sets. The membership functions of $S\cap T$, $S\cup T$, and $\overline{S}$ were defined as follows:

$\mu_{S\cap T}(x)=\min(\mu_S(x),\ \mu_T(x))$,

$\mu_{S\cup T}(x)=\max(\mu_S(x),\ \mu_T(x))$,

$\mu_{\overline{S}}(x)=1-\mu_S(x)$.

A fuzzy relation R is defined as a fuzzy collection of ordered pairs. We treat here a special case of the fuzzy relation. That is, we consider similarity relations introduced in [7]. The similarity is essentially a generalization of the notion of equivalence, and it is very useful concept for the theory of knowledge representation.

Definition 2.3 A similarity relation R,, in X, is a fuzzy relation in X which satisfies the following (a)-(c):

(a) reflexive, $\mu_R(x,x)=1$ for all x in dom R,

(b) symmetric, $\mu_R(x,y)=\mu_R(y,x)$ for all x,y in dom R,

(c) transitive, $\mu_R(x,z)\geqslant \bigvee_y (\mu_R(x,y)\wedge\mu_R(y,z))$ for all x,y,z in dom R.

<u>Definition 2.4</u> For α in [0,1], an α-level set of a fuzzy set S in X is a non-fuzzy set defined as $S_\alpha=\{x \in X \mid \mu_S(x) \geq \alpha\}$.

<u>Definition 2.5</u> For α in [0,1], an α-level set of a fuzzy relation R is a non-fuzzy subset of X^2 defined as $R_\alpha=\{(x,y) \mid \mu_R(x,y) \geq \alpha\}$.

<u>Definition 2.6</u> For x of a non-fuzzy set X, a fuzzy set S, $\alpha \in [0,1]$ and a similarity R, $\mu_{\widetilde{R}_\alpha(S)}(x)$ and $\mu_{\underset{\sim}{R}_\alpha(S)}(x)$ are defined as follows:

$$\mu_{\widetilde{R}_\alpha(S)}(x) = \sup_{\mu_R(x,x') \geq \alpha} \mu_S(x'),$$

$$\mu_{\underset{\sim}{R}_\alpha(S)}(x) = \inf_{\mu_R(x,x') \geq \alpha} \mu_S(x') .$$

Notice here that $\widetilde{R}_\alpha$ and $\underset{\sim}{R}_\alpha$ correspond to R^* and R_* of rough sets, respectively.

§3 Fuzzy rough logic

In this section, we propose a logic based on fuzzy rough sets. First, we give the syntax. In this logic, we use the following symbols:

(1) Propositional variables: $X,Y,Z,\ldots,X_1,X_2,\ldots$,

(2) Logical symbols: $\wedge$ (and), $\vee$ (or), $\neg$ (not),

(3) Similarity operations: R_α, $R_\beta,\ldots$, S_α, ... ($\alpha,\beta \in [0,1]$),

(4) Parentheses: (,) .

<u>Definition 3.1</u> A well-formed formula (for short wff) is defined as follows:

(1) Propositional variables are wff's.

(2) Let A and B be wff's. Then, $(A)\wedge(B)$, $(A)\vee(B)$, $\neg(A)$, and $\widetilde{R}_\alpha(A)$ are wff's.

(3) A wff is defined by the above (1) and (2) only.

In (2) of Definition 3.1, parentheses (,) are abbreviated in the usual rules. Further, $\underset{\sim}{R}_\alpha$ is defined as $\neg\tilde{R}_\alpha\neg$.

Example $(\tilde{R}_{0.5}(X \vee Y) \wedge \underset{\sim}{S}_{0.7}(X \vee \neg Z)) \vee \tilde{R}_{0.1}\neg X$ is a wff.

The proposed fuzzy rough logic is denoted by FRL. Let us now consider the semantics of FRL. Let L be the set of propositional variables and OB is the set of objects. A *model* is a triple $\langle OB, \{R_\nu\}, \mu\rangle$.

Definition 3.2 An α-satisfiability of a wff A is defined as follows:

There exists an object x in a model $M=\langle OB, \{R_\nu\}, \mu\rangle$ such that

(1) The case that A is a propositional variavle X:

Considering X as a fuzzy set, X is α-satisfiable iff $\mu_X(x) \geq \alpha$.

In the following cases, propositional variables are always considered as fuzzy sets.

(2) The case that A is a wff $A_1 \vee A_2$, or $A_1 \wedge A_2$, or $\neg A_1$:

$A_1 \vee A_2$ is α-satisfiable iff $\mu_{A_1 \cup A_2}(x) \geq \alpha$

$A_1 \wedge A_2$ is α-satisfiable iff $\mu_{A_1 \cap A_2}(x) \geq \alpha$

A_1 is α-satisfiable iff $\mu_{\bar{A}_1}(x) \geq \alpha$.

(3) The case that A is a wff $\tilde{R}_\beta A_1$:

$\tilde{R}_\beta A_1$ is α-satisfiable iff $\mu_{\tilde{R}_\beta A_1}(x) \geq \alpha$.

When a wff A is α-satisfiable at x of M, this is denoted by

$$M, x \models_\alpha A \quad .$$

Example

Every wff is 0-satisfiable.

$X \wedge \neg X$ is not 0.6-satisfiable.

$\tilde{R}_{0.3}X$ is 0.5-satisfiable.

§4 Decision problem

Here, we give a result of a decision problem on the logic FRL. That is, we show a positive solution to decide whether or not an arbitrary wff of the logic based on fuzzy rough sets is α-satisfiable.

Lemma 4.1 The following (1)-(4) hold.

(1) $\mu_{F \vee G}(x) \geq \alpha \Leftrightarrow \mu_F(x) \geq \alpha$ or $\mu_G(x) \geq \alpha$.

(2) $\mu_{F \wedge G}(x) \geq \alpha \Leftrightarrow \mu_F(x) \geq \alpha$ and $\mu_G(x) \geq \alpha$.

(3) $\mu_{\neg F}(x) \geq \alpha \Leftrightarrow \mu_F(x) \leq 1-\alpha$.

(4) $\mu_{\tilde{R}_\beta F}(x) \geq \alpha \Leftrightarrow \exists y((x,y) \geq R_\beta$ and $\mu_F(y) \geq \alpha$).

Proof

These (1)-(4) are easily shown from the definitions. //

Lemma 4.2 Let A be a wff of FRL. Then, $\mu_A(x) \geq \alpha$ is transformed to a formula $A^{¢}$ in which μ-operations are just before propositional variables and $\mu_A(x) \geq \alpha \Leftrightarrow A^{¢}$ is satisfiable at x of M.

Proof

This is obvious from Lemma 4.1 . //

Generally, $A^{¢}$ means a wff having μ-operations just before propositional variable, which is obtained by repeated applications of (1)-(4) of Lemma 4.1 from a wff A of FRL.

Now, we consider the problem to decide whether or not an arbitrary wff A is α-satisfiable. Hereafter, we assume that α is an arbitrary but fixed number. First, we can assume that OB for A is infinite. Because for a finite case the α-satisfiability of A is decided by checking all cases.

From Lemma 4.2, $A^{¢}$ can be also considered as a wff of the first-order predicate logic which consists of some duadic predicate letters $R_\alpha(x,y)$, $S_\beta(z,x)$,... and monadic predicate letters $X_\alpha(x)$, $Y_\gamma(u)$,.... According to this consideration, we define the equivalence class over OB. Let M be a model and $S(A^{¢})$ be the set consisting of all formula $\Phi^{¢}$'s which are obtained from subformulas Φ's of A.

Definition 4.3 For x,y in OB, $x \approx y$ $(M,A^{¢})$ (for short, $x \approx y$ in case that there is no confusion) is defined as follows:

$$x \approx y \quad \text{iff} \quad M,x \models \Phi^{¢} \Longleftrightarrow M,y \models \Phi^{¢} \quad \text{for all } \Phi^{¢} \in S(A^{¢}),$$

and

$M,x \models (\underset{\sim}{R}_\beta \Phi)^{¢} \Leftrightarrow M,y \models (\underset{\sim}{R}_\beta \Phi)^{¢}$ for all R_β's in A and all subformulas Φ's of A,

and

$M,x \models (\underset{\sim}{R}_\beta \underset{\sim}{S}_\gamma \Phi)^{¢} \Leftrightarrow M,y \models (\underset{\sim}{R}_\beta \underset{\sim}{S}_\gamma \Phi)^{¢}$ for all R_β's and S_γ's in A and all subformulas Φ's of A.

It is easily known from the definition that $\approx$ is an equivalence relation. Thus, [x] defined as $[x]=\{x' \mid x \approx x'\}$ is an equivalence class. Also, it is known that OB is divided into finite equivalence classes by this relation. This is denoted by $OB^{\#}$.

By making use of this [x], the relation $R^{\#}_\beta$ over $OB^{\#} \times OB^{\#}$ is defined as follows:

$([x],[y]) \in R^{\#}_\beta$

iff

$M,x \models (\underset{\sim}{R}_\beta \Phi)^{¢} \Leftrightarrow M,y \models (\underset{\sim}{R}_\beta \Phi)^{¢}$ for all R_β's in A and all subformulas Φ's of A,

and

$M,x \models (\underset{\sim}{R}_\beta \underset{\sim}{S}_\gamma \Phi)^{¢} \Leftrightarrow M,y \models (\underset{\sim}{R}_\beta \underset{\sim}{S}_\gamma \Phi)^{¢}$ for all R_β's and S_γ's in A and all subformulas Φ's of A.

It is easily known from the definition of [x] that this relation $R^{\#}_\beta$ is well defined. Then, $M^{\#}$ is defined by $\langle OB^{\#}, \{R^{\#}_\nu\}, \mu^{\#} \rangle$. It is shown without difficulty that $M^{\#}$ is a model.

<u>Theorem 4.4</u> (the main theorem)

$M,x \models_\alpha A$ iff $M,[x] \models_\alpha A$.

Proof

We prove this theorem for a case that A is of form $\underset{\sim}{R}_\beta A_1$. For cases that A is of form $A_1 \vee A_2$ or $A_1 \wedge A_2$ or $\neg A_1$, it is easily shown by defining $\mu^{\#}_X([x])$ as $\mu_X(x)$.

In the following (i) and (ii), α is a fixed number of [0,1]. We prove (i) and (ii) by the induction of number of $\vee, \wedge, \neg, R_\nu$ in A.

(i) Let us assume that $\underset{\sim}{R}_\beta A_1$ is α-satisfiable at [x] in a model $M^\#$. The base of induction is obvious by defining $\mu^\#$. The induction step is shown as follows: It is easily shown that $(x,y)\in R_\beta$ implies $([x],[y])\in R_\beta^\#$. Thus, we know that $\forall[y]\ (([x],[y])\in R_\beta^\# \supset \mu_{A_1}([y])\geq\alpha)$ implies $\forall y((x,y)\in R_\beta \supset \mu_{A_1}([y])\geq\alpha)$. Therefore, we have $M,x\models_\alpha \underset{\sim}{R}_\beta A_1$ from the hypothesis of induction.

(ii) From $M,x\models_\alpha \underset{\sim}{R}_\beta A_1$ we must $M,[x]\models_\alpha \underset{\sim}{R}_\beta A_1$. The base of induction is also ovbious. For the proof, it is sufficient to show

$$\forall y((x,y)\in R_\beta \supset \mu_{A_1}(y)\geq\alpha) \Rightarrow \forall[y]\,(([x],[y])\in R_\beta^\# \supset \mu_{A_1}([y])\geq\alpha).$$

But, this is shown by making use of the following fact and the hypothesis of induction.

$$\forall y((x,y)\in R_\beta \supset \mu_{A_1}(y)\geq\alpha)\ \&\ (M,x\models(\underset{\sim}{R}_\beta A_1)^¢ \Leftrightarrow M,y\models(\underset{\sim}{R}_\beta A_1)^¢) \Rightarrow M,y\models_\alpha A_1 .$$

From the above (i) and (ii), we get this theorem. //

From Theorem 4.4, we have the following corollary:

<u>Corollary 4.5</u> The decision problem to decide whether or not an arbitrary wff A of FRL is α-satisfiable is solvable.

Proof

From Theorem 4.4, we can determine a cardinal number of $OB^\#$ which is finite and depends on a given wff A.

Thus, we get this corollary. //

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